. Curs 12

CARDINAL NUMBERS

18.12.2018

e have the same power)

We explain what is the number of elements of a set

Def. The sets A and B are equipotent (not A~B) if I a bijective function FIA →B.

Lemma : The equipotence relation "" is an equivalence relation

proofi (R) A~A because the identity map MA:AJA, MA (a)=a, faca

is bijective. (T) Assume A~B and B~e.

Then I fi ADB and g: Boe bijective functions, we know that the map gof: ADC is also bijı, hence

ANC. (5) Assume that A~B. Then I fi ADB bijective. We know.

that ft: B-A is also bejective, hence B~A.

. Deti The cardinal number of the set A is the equipotence class of A, ine, | 18+ get RB A 8.

Remarki ) This is the "naie" definition given by Georg Cantor ~ 1870

2) An B <=> 141 = 1B1 Chave the same cardinality )

Operations with cardinal numbers; 1) Addition: Idea; if Anb=0, then I AL HIBI def LAUBI

In general, let (xi)iet be a family of cardinal numbers, where a;= 1 Ail, so the set A is a representative for Xi. The problem is that the sets Ai; ie I may not be pairwise disjoint,

" We replace the family (Ailie I with another family (A'Dict, where Ai def Ai x {i} = {(ai, i) lai € Ai?. Obviously Ai~ A'i and Arn A'; = 0 if iti

Def! E ai deg 4'11 = 1 y Aix {711 Remark! The definition does not depend on the choice of representatives

i 385

2) Multiplication : *Idea* : 3x5 E

Let (ai) i el be a family of cardinal numbers

Defi mai = lite Ail, where II Ai := {(ai) ret lai e Ai Vie I? is the generalized cartesian product of the *f*a*mily (M*iliei C&i=*1 ti*ll.

*R*emark: The *de*finition does not *d*epend on *t*he choice *of* repres*ent*atives

iel

3) Exponentiation :

Let a = 1 Al and B=1B).

Defi B9\_ IBA) = | Hom (A,B), where BA = Hom (A,B)={fl fia¬B? is the set of functions from A to B.

Remark! The definition does not *d*epend on the choice o*f r*epresentatives

Theorem ( the properties of the operations ) :: 1) Addition and multiplication are commutateve 2) Addition and multiplication are asscciative 3) The multiplication is distributive wir.t, addition

(j)elef

(Emi). (R3) = Skip 4).p3" - I poolt

Ala

6) Jas (58)

proofi 18,2) (we skip them).

3) Let (i = 1 Ail, Bj = 1 *B*; /

(EX) (E Rj) - 1 ] (A; x {i}) x (Bj \*{}}]]

-}(2,1), (b);j))/ ai chi ie I, bj e Bj, je I} I xißj = || (A; x Bj) \* { (ij)} |

jtd

Lijje Ix]

"Lije ix]

*= } li a,b*), (i)))/ ai e di, bj e Bir Cijs e Ix]*}* We need a bijective function |

UCA: x{1}) \* U (B x {j}) e Ai XB;) = {(1))} TEI

Obviously (Cai, i), (bj. j)) ? | cai, b;), (1, j')) is a bijection, because it has an obvious inverse.

proof : 4),5) - use the "universal property of the direct sum and direct.

product "

6) Let at IAI, B=1B1, 5=iCl.

Then J AB = | Hom CAXB,C)|.

(8-3)\* = | Hom (A, Hom CB,C)) | In order to prove the equality, we must find bijective functions:

Hom (AxB, C) = Q Hom (A, Hom (B,C)) s.t. 4= 6-1

"Let f: Axb-DC. We want to define P(f): A) Hom (B,C),

So Ylf)(a) : B > *C* so p (f) (a) (b) EC We define 14(f) (a) (b) def flaib)/ olet 9:45 Hom (B,C), so gea) : B-)C, so grascb) E C fact, bet

We want to define 4(q): AxB-C: so let 14 (g) (a, b) det gralcb)...

The above definitions show that *$CA)=9*0) 469)=f, hence

Y=64; so p is bijective . Prop: 141+IBI = LAUBI + I AMBI

LANBRIA

AUB = *AU(BVA*) c disjoint)

hence IAUBLE I AL FIBVAL. => 1 AUBIA LANBI=*1*41+IBIAI PIANO But we have B = (BVA) U*CANB)*,

( disjoint union ) hence IBL = IBVAIHIAMBI.

(\*) = 141+iBl

Theorem (Cantor): | PCA)) = 21A). Proof! P(A) = {X/ XCA?

and 2 "A!= | Hom (A, 20,13), so we must find bijective functions, e, 4 where Y= 4-1

P(A) Hom (A, 20,13)

Let Xx: A > {0,14 he the characteristic function of the

2x cal def 21, if a ex subset X of A

B lo, if a 4x Let 4(X) def Yx E Hom (H, 20,1%) Let 4 (X) def X^(1) = {a e Alx (a) = 1? for any X: A) {0,13. We have (404)(x) = 4(4(XJ) - Mix CM) = X = 1(%); (404)(x) = 4.4(X)) = 4(X^(1)) = 2x+60 = x - 16X)

Ordering cardinal numbers Defi Let a= lal , B=IB).

. Then asb (def F fi ADB an injective function Rem: the definition *d*oes not depend on the choice o*f r*epresentatives

Lemma. The relation "s" is an order relation.

proofi (R) as a because IA: ASA is injective

(T) Assume ass and B5 8. Let fi AB and gi BPC inji

We know that gofi Ade is also injective; hence asr (A) Assume asp and psa. Let fi A-JB and g: B-JA

be injective functions

We must show that I h: AJB bijective (not easy!). It is a - consequence of the Cantor- Bernstein - Schr*öd*er theorem.

Theorem ( Cantor). *d*<24

proofi Let x=lAl, So 2\_ | PCA))

We have an injective function f: A- PCA), fra)= {Q? hence IAIS IPCA)) Assume, by contradiction, that I a bijective function Pi A- PCA). Let X = hae Ala 44(a) 4 E PCA) By assumption , XEA s.t. (%) = X

Case 1: Assume xex. Then xe ecx) => x does not satisfy the

condition in the def of X -> x¢ X contrad*i*ction Case 2: *A*ssume x® X =) X *& pc*x) =) x satisfies the condition

in the def of X = Xe X contradiction Hence, we do not have bjective functions A -> P*CA*).